3.1 Exponential Functions and Their Graphs

Two types of **nonalgebraic functions** are **EXPONENTIAL** and **LOGARITHMIC FUNCTIONS**.

Both of these functions are examples of **Transcendental Functions**; a function that cannot be expressed in terms of a finite sequence of the algebraic operations additions, multiplications, and root extraction.

EXPONENTIAL FUNCTION: $f(x) = a^x$ ← the variable is the exponent.

The function f with base a; where $a > 0$, $a \neq 1$, and x is any real number.

Ex. 1: Simplify each expression (Give exact values; remember exponent rules).

a) $8^{\sqrt{6}} \cdot 4^{\sqrt{54}}$

$(2^3)^{\sqrt{6}} \cdot (2^2)^{\sqrt{54}}$

$2^{3\sqrt{6}} \cdot 2^{2\sqrt{54}}$

$2^{3\sqrt{6}} \cdot 2^{2 \cdot 3\sqrt{6}}$

$2^{3\sqrt{6}} \cdot 2^{6\sqrt{6}}$

$2^{9\sqrt{6}}$

54 → 2 · 27, 27 → 3 · 9, 9 → 3 · 3

b) $(6^{\sqrt{5}})^{\sqrt{6}}$

$6^{\sqrt{30}}$

c) $5^{\sqrt{3}} \div 5^{\sqrt{2}}$

$\frac{5^{\sqrt{3}}}{5^{\sqrt{2}}} \rightarrow 5^{\sqrt{3}-\sqrt{2}}$

Ex. 2: Evaluate (round to four decimal places)

a) $3^{2.6}$ ≈ 17.3986

b) $3^{-2.9}$ ≈ 0.0413

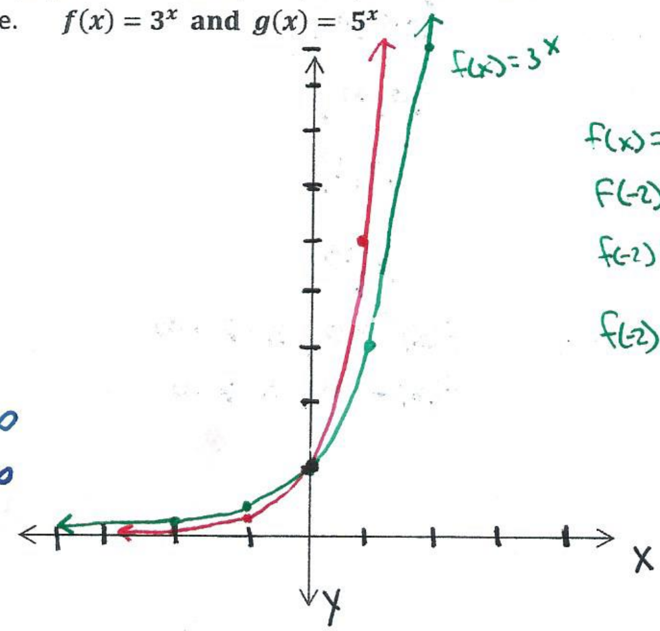
c) $(.81)^{\frac{4}{5}}$ $\approx .8449$

d) $3^{\sqrt{7}}$ ≈ 18.2955

e) 5^{π} ≈ 156.9925

Ex. 3: Complete the table and graph each function on the graph provided. Identify the y-int., x-int., state the end behavior, find any HA, and state the domain and range. $f(x) = 3^x$ and $g(x) = 5^x$

x	-2	-1	0	1	2
f(x)	1/9	1/3	1	3	9
g(x)	1/25	1/5	1	5	25



$f(x) = 3^x$

$f(-2) = 3^{-2}$

$f(-2) = \frac{1}{3^2}$

$f(-2) = \frac{1}{9}$

y-int	x-int	End-behavior
(0,1)	None	$f(x) \to 0, x \to -\infty$; $f(x) \to \infty, x \to \infty$

HA	Domain	Range
$y = 0$	$(-\infty, \infty)$	$(0, \infty)$

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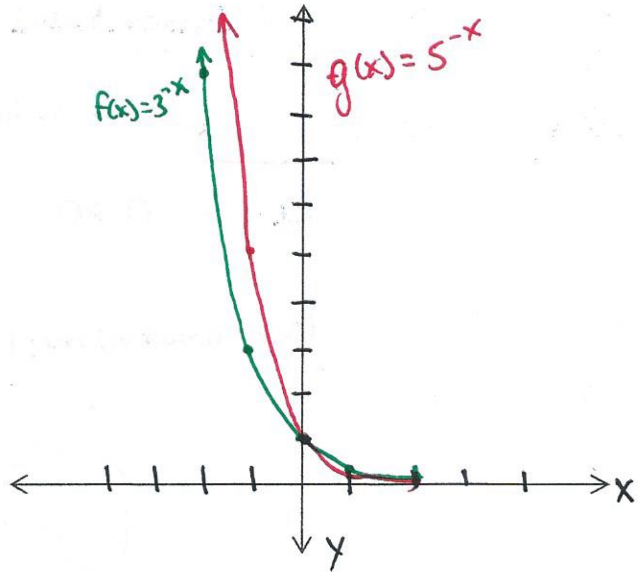
Ex. 4: Complete the table and graph each function on the graph provided. Identify the y-int., x-int., state the end behavior, find any HA, and state the domain and range. $f(x) = 3^{-x}$ **and** $g(x) = 5^{-x}$

x	-2	-1	0	1	2
f(x)	9	3	1	1/3	1/9
g(x)	25	5	1	1/5	1/25



y-int: (0,1)

x-int: None

End-behavior:
$f(x) \to \infty, x \to -\infty$
$f(x) \to 0, x \to \infty$

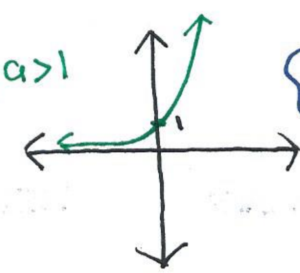
HA: $y=0$

Domain: $(-\infty,\infty)$

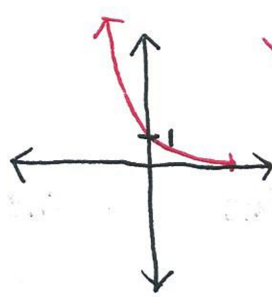
Range: $(0,\infty)$

Graphs of:

$y=a^x; a>1$



reflections across the y-axis



$y=a^{-x}; a>1$

	$y=a^x; a>1$	$y=a^{-x}; a>1$
Domain:	$(-\infty,\infty)$	$(-\infty,\infty)$
Range:	$(0,\infty)$	$(0,\infty)$
y-int:	(0,1)	(0,1)
HA:	$y=0$	$y=0$
End Behavior:	$f(x) \to 0, x \to -\infty$ $f(x) \to \infty, x \to \infty$	$f(x) \to \infty, x \to -\infty$ $f(x) \to 0, x \to \infty$

Continuous

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Ex. 5: Apply the indicated operation to the function $f(x) = 5^x$. Describe the change, state the y-int., and the HA. Check your answers using a calculator.

a) $f(x+1)$

$g(x) = 5^{x+1}$

Shifts left 1

y-int: (0,5)

HA: y=0

b) $f(x) - 2$

$g(x) = 5^x - 2$

Shifts down 2

y-int: (0,-1)

HA: y=-2

c) $-f(x)$

$g(x) = -5^x$

reflects over x-axis

y-int: (0,-1)

HA: y=0

d) $f(-x)$

$g(x) = 5^{-x}$

reflects over y-axis

y-int: (0,1)

HA: y=0

Two types of Exponential Functions:

Exponential Growth; if $a > 0$ and $b > 1$ then $y = ab^x$

Exponential Decay; if $a > 0$ and $0 < b < 1$ then $y = ab^x$

a = initial amount

b = growth / decay factor

SAME FUNCTION EXCEPT FOR VALUE OF "b"

Ex. 6: Determine whether each function represents exponential GROWTH or DECAY. State the growth/decay factor and the initial amount.

a) $y = 4(\frac{1}{2})^x$

decay; 1/2

4 initial

b) $y = \frac{1}{3}(7)^x$

growth; 7

1/3 initial

c) $y = 2.5^x$

growth; 2.5

1 initial

d) $y = 5(7/3)^x$

growth; 7/3

5 initial

Ex. 7: Write the exponential function whose graph passes through the given points.

* $y = ab^x$ (y-int is the initial amount) *

a) (0, 1) & (-1, 4)

$1 = ab^0$

$a = 1$

$4 = 1(b)^{-1}$

$4 = b^{-1}$

$4 = \frac{1}{b}$

$b = 1/4$

$\boxed{y = 1(1/4)^x}$

b) (0, 2) & (1, 10)

$a = 2$

$10 = 2(b)^1$

$10 = 2b$

$b = 5$

$\boxed{y = 2(5)^x}$

c) (0, 7) & (2, 63)

$a = 7$

$63 = 7(b)^2$

$9 = b^2$

$b = 3$

$\boxed{y = 7(3)^x}$

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Ex. 8:

In 1983 there were 102,000 farms in North Carolina, but by 1998 this number had dropped to 80,000.

Start time   initial amount

a) Write an exponential function to model the farm population y, (of NC); x is the number of years since 1983. (round to four decimal places) Let 1983 = 0, 1984 = 1, ...

1998
−1983
15 yr

(0, 102,000)
(15, 80,000)

$80000 = 102000(b)^{15}$

$.7843 = b^{15}$

$b \approx .9839$

$y = 102{,}000(.9839)^x$

b) Suppose the number of farms continues to decrease at the same rate. Estimate the number of farms in 2010.

$y = 102{,}000(.9839)^x$

2010
−1983
27

$y = 102{,}000(.9839)^{27}$

$y \approx$ 65,867 farms

Ex. 9: The following table represents the historical prices of gold from 1991 to 2008.

1991	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001
379.9	356.3	419.2	409.8	385.6	367.8	288.8	288	287.5	272.15	278.7

Let 1991 = 0, 1992 = 1, 1993 = 2....

2002	2003	2004	2005	2006	2007	2008
346.7	414.8	438.1	517.2	636.3	833.2	904.8

a) Write the equation of the exponential function by hand.

(0, 379.9)
(1, 356.3)

$356.3 = 379.9(b)^1$

$b \approx .9379$

$y = 379.9(.9379)^x$

b) Using a calculator, write the exponential regression equation. State the r value (4 decimal places)

$y = 290.9651(1.0402)^x$ $r = .5936$

c) Is the equation from (b) an accurate representation of the data?

No, the r-value is not close to 1. The trend line does not follow the data well.

d) Is the equation in (a) an accurate representation of the data? Why?

No, the trend line shows exponential decay but data shows growth. The trend line does not follow the data.

e) Which is a better representation of the data (a) or (b)? Why?

b, the calculator regression shows a trend line with growth.

3.1 Exponential Functions and Their Graphs

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Now use the following table below to answer "a-e" above.

0	1	2	3	4	5
60	85	100	122	139	156

a) (0,60) $85 = 60(b)^1$ $y = 60(1.4167)^x$

(1,85) $b \approx 1.4167$

b) $y = 66.3032(1.2024)^x$; $r = .9808$

c) Yes, the r-value is .98; close to 1. The trend line follows the data closely.

d) NO, it starts off well but as X increases, the trend line curves up more sharply than the data.

e) The calculator's exponential regression!

Base e^

Is a symbol that represents a value of approximately 2.71828.

e is called the Natural base and the function $f(x) = e^x$ is called the Natural Exponential.

Domain: $(-\infty, \infty)$ Range: $(0, \infty)$ y-int: (0,1)

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Ex. 10: Evaluate the expression $\left(1+\frac{1}{x}\right)^x$ with the given values below. (four decimal places)

x	10	100	1000	10,000	100,000	1,000,000
f(x)	2.5937	2.7048	2.7169	2.7181	2.7183	2.7183

$x \to \infty$, $f(x) \to e$

Ex. 11: Evaluate to four decimal places.

a) e^{-4} $\approx .0183$

b) e^3 ≈ 20.0855

c) $e^{2\pi}$ ≈ 535.4917

Ex. 12: (Look at page 221 charts and graphs)

Compound Interest: $A = P\left(1+\frac{r}{n}\right)^{nt}$

Example of exponential growth

If the interest is added to the principal at the end of the year the balance is $P_1 = P + Pr$ or $P_1 = P(1 + r)$

Time (yrs)	Balance after each compound
0	$P = P$
1	$P_1 = P(1+r)$
2	$P_2 = P(1+r)^2$
3	$P_3 = P(1+r)^3$
…	
n	$P_n = P(1+r)^n$ ← Applies only for compounded ANNUALLY

If the principal is \$1000 and the rate is 2%, what is the amount after 5 years?

$A = 1000(1+.02)^5$ $A = \$1104.08$

Types of Compounded Periods:

n
- Quarterly 4
- Monthly 12
- Semi-annually 2
- Weekly 52
- Daily 365

* Use $\frac{r}{n}$ and nt if not compounded annually. *

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Ex. 13:

Jim invests \$500 at 8% interest compounded quarterly. How much does he have after 10 years?

$A = P(1+\frac{r}{n})^{nt}$ $A = 500\left(1+\left(\frac{.08}{4}\right)\right)^{4(10)}$

$A = \$1,104.02$

If you let the number of compounds, "n", increase without bound (without limits), you approach continuous Compound.

$A = Pe^{rt}$ used for compounding continuously.

natural base

Ex. 14:

A total of \$2500 is invested at an annual interest rate of 6%. Find the balance after 10 years if it is compounded:

a) Annually b) Quarterly c) Monthly d) Continuously

$A = P(1+\frac{r}{n})^{nt}$ $A = Pe^{rt}$

$A = 2500(1+.06)^{10}$ $A = 2500(1+\frac{.06}{4})^{4(10)}$ $A = 2500(1+\frac{.06}{12})^{12(10)}$ $A = 2500e^{.06(10)}$

$A = \$4477.22$ $A = \$4,535.05$ $A = \$4548.49$ $A = \$4555.30$

Ex. 15:

Let y represent the mass of a quantity of a radioactive element whose half-life is 30 years. After t years, the mass (in grams) is $y = 14\left(\frac{1}{2}\right)^{\frac{t}{30}}$.

a) What is the initial mass (when t=0)?

$y = 14(1/2)^0$ $y = 14$ grams

b) How much of the initial amount is present after 75 years?

$y = 14(1/2)^{75/30}$ divide by 30 b/c of a 30 yr half-life. So in 75 yrs this would be 2.5 half-lives.

$y = 14(1/2)^{2.5}$

$y \approx 2.47$ grams

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Ex. 16:

The approximate number of bacteria in an experimental population after t hours is $B(t) = 15e^{.02t}$.

a) What is the initial number of bacteria?

$B(t) = 15e^{.02(0)}$ * time = 0 for initial amount *

$B(t) = 15$ bacteria

b) After 72 hours, what is the population size?

$B(t) = 15e^{.02(72)}$

$B(t) = 63$ bacteria   * round down b/c you can not have partial bacteria.

c) Graph in calculator