9.1 Sequences and Series

Wednesday, April 29, 2015
10:23 AM

Sequence

Is a collection that is **ORDERED**.

- ordered from first to last
- is a function whose domain is the Set of Positive Integers.
- written with subscript notation ($a_n = 2n+1$)

Infinite Sequence

A function whose domain is the set of positive integers. The function values: $a_1, a_2, a_3, a_4, \dots, a_n$ are the terms of the sequence. If the domain of the function consists of the first n positive integers **ONLY**, the sequence is called Finite Sequence.

Ex. 1 Find the first six terms of each sequence.

a) $a_n = 3n-2$

$a_1 = 3(1)-2 = 1$ $\ +3$

$a_2 = 3(2)-2 = 4$ $\ +3$

$a_3 = 3(3)-2 = 7$ $\ +3$

$a_4 = 3(4)-2 = 10$ $\ +3$

$a_5 = 3(5)-2 = 13$ $\ +3$

$a_6 = 3(6)-2 = 16$

b) $a_n = \frac{n-1}{n}$

$a_1 = \frac{1-1}{1} = 0$

$a_2 = \frac{2-1}{2} = 1/2$

$a_3 = \frac{3-1}{3} = 2/3$

$a_4 = 4-1/4 = 3/4$

$a_5 = 5-1/5 = 4/5$

$a_6 = 6-1/6 = 5/6$

c) $b_n = (-1)^{n-1}\left(\frac{2}{n}\right)$

d) $c_n = \begin{cases} n & \text{if } n \text{ is even} \\ \frac{1}{n} & \text{if } n \text{ is odd} \end{cases}$

$b_1 = 2$
$b_2 = -1$
$b_3 = 2/3$
$b_4 = -1/2$
$b_5 = 2/5$
$b_6 = -1/3$

$c_1 = 1$
$c_2 = 2$
$c_3 = 1/3$
$c_4 = 4$
$c_5 = 1/5$
c_6 6

In each example above, you were given the "n^{th} term", which is the formula. The listing of a few terms (like above) is not **SUFFICIENT** to define a sequence, the "n term" must be written.

Ex. 2 Determine the n^{th} term of each sequence.
* YOU HAVE TO THINK! LOOK FOR PATTERNS!

a) $1, 3, 5, 7, \ldots$

$a_1 = 1$ $>+2$
$a_2 = 3$ $>+2$
$a_3 = 5$ $>+2$
$a_4 = 7$
$a_n = 2n-1$

n =	1	2	3	4
terms:	1	3	5	7

b) $2, 5, 10, 17, \ldots$

$a_1 = 2$ $>+3$
$a_2 = 5$ $>+5$
$a_3 = 10$
$a_4 = 17$
$a_n = n^2+1$

c) $1, 1/3, 1/9, 1/27$

$a_n = \dfrac{1}{3^{n-1}}$

d) $e, \dfrac{e^2}{2}, \dfrac{e^3}{3}, \dfrac{e^4}{4}$

$a_n = \dfrac{e^n}{n}$

When signs of a sequence alternate, you use factors such as: $(-1)^{n+1}$ which equals 1 if n is odd and -1 if n is even

$(-1)^n$ which equals -1 if n is odd and 1 if n is even

$1, -1/2, 1/3, -1/4, 1/5$ $\quad a_n = (-1)^{n+1}(1/n)$

$-1, 1/2, -1/3, 1/4, -1/5$ $\quad a_n = (-1)^n (1/n)$

Recursive Sequence

- need to be given one or more of the first few terms.
- all other terms are defined using previous terms.

ie Fibonacci Sequence

$1,1,2,3,5,8,13,\ldots \longrightarrow a_k = a_{k-2} + a_{k-1}$

Terms 1 2 3 4 5 6 7

$a_3 = a_{k-2} + a_{k-1}$
$= a_{3-2} + a_{3-1}$
$= a_1 + a_2$
$= 1 + 1$
$a_3 = 2$

$a_8 = a_{8-2} + a_{8-1}$
$= a_6 + a_7$
$= 8 + 13$
$a_8 = 21$

Ex.3 Write the first 5 terms of the following recursive sequences.

a) $s_1 = 1$; $s_n = 4s_{n-1}$

$s_2 = 4s_{2-1} \rightarrow 4s_1 \rightarrow 4(1) = 4$
$s_3 = 4s_{3-1} \rightarrow 4s_2 \rightarrow 4(4) = 16$
$s_4 = 4s_{4-1} \rightarrow 4s_3 \rightarrow 4(16) = 64$
$s_5 = 4s_{5-1} \rightarrow 4s_4 \rightarrow 4(64) = 256$

b) $f_1 = 1$; $f_n = nf_{n-1}$

$f_2 = 2f_{2-1} \rightarrow 2f_1 \rightarrow 2(1) = 2$
$f_3 = 3f_{3-1} \rightarrow 3f_2 \rightarrow 3(2) = 6$
$f_4 = 4f_{4-1} \rightarrow 4f_3 \rightarrow 4(6) = 24$
$f_5 = 5f_{5-1} \rightarrow 5f_4 \rightarrow 5(24) = 120$

Factorial (#!)

A special type of Product

Special case

$0! = 1$

$1! = 1$

$n! = 1\cdot 2\cdot 3\cdot 4\cdot \ldots (n-1)\cdot n$

$n! = n(n-1)(n-2)(n-3)\ldots \quad n \geq 2$

$3\cdot 5!$

$0! = 1$

$1! = 1$

$2! = 1(2) = 2$

$3! = 1(2)(3) = 6$

$4! = 1(2)(3)(4) = 24$

$5! = 1(2)(3)(4)(5) = 120$

$3\cdot 5! = 3(1\cdot 2\cdot 3\cdot 4\cdot 5)$

$= 3(120)$

$= 360$

Factorials follow the same conventions for order of operations as do exponents.

Ex. 4 Write the first 5 terms; begin with $n=0$!

a) $a_n = \frac{2^n}{n!}$

$a_0 = \frac{2^0}{0!} = \frac{1}{1} = 1$

$a_1 = \frac{2^1}{1!} = \frac{2}{1} = 2$

$a_2 = \frac{2^2}{2!} = \frac{4}{2} = 2$

$a_3 = \frac{2^3}{3!} = \frac{8}{6} = \frac{4}{3}$

$a_4 = \frac{2^4}{4!} = \frac{16}{24} = \frac{2}{3}$

Ex. 5 Simplify (show your work)

a) $\dfrac{7!}{3!4!}$

$$\frac{7\cdot\cancel{6}\cdot5\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{(\cancel{3}\cdot\cancel{2}\cdot1)(\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1})}=\frac{7\cdot5}{1}$$

b) $\dfrac{4!5!}{2!3!}$

$$\frac{(4\cdot3\cdot\cancel{2}\cdot\cancel{1})(5\cdot4\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1})}{(\cancel{2}\cdot\cancel{1})(\cancel{3}\cdot\cancel{2}\cdot1)}=4\cdot3\cdot5\cdot4$$

{35} {240}

c) $\frac{(n+8)!}{(n+2)!}$

$$\frac{(n+8)(n+7)(n+6)(n+5)(n+4)(n+3)\cancel{(n+2)!}}{\cancel{(n+2)!}}$$

$(n+1)(n)(n-1)(n-2)(n-3)$

$(n+8)(n+7)(n+6)(n+5)(n+4)(n+3)$

d) $\frac{(n-4)!}{n!}$ $\qquad \frac{\cancel{(n-4)!}}{n(n-1)(n-2)(n-3)\cancel{(n-4)!}}$

$(n-5)(n-6)(n-7)\ldots$

$$\frac{1}{n(n-1)(n-2)(n-3)}$$

Series

The addition of the terms of a sequence.

ie Sequence: 1, 2, 3, 4, 5, ...
   Series: 1+2+3+4+5+ ....

Summation Notation (Sigma Notation)

$$\sum_{K=1}^{n} a_K \qquad \qquad \sum_{K=1}^{3} K = 1+2+3 = 6$$

3 ← END

K=1 ← Start

K is the index; tells you where to start!
n is where you end!

[illegible]

d) $\dfrac{(n-4)!}{n!}$ $\qquad \dfrac{\cancel{(n-4)}!}{n(n-1)(n-2)(n-3)\cancel{(n-4)}!}$ $\qquad (n-5)(n-6)(n-7)\ldots$

$$\frac{1}{n(n-1)(n-2)(n-3)}$$

Series

The addition of the terms of a sequence.

i.e. Sequence: 1, 2, 3, 4, 5, ...
Series: 1 + 2 + 3 + 4 + 5 + ...

Summation Notation (Sigma Notation)

$$\sum_{k=1}^{n} a_k \qquad \sum_{k=1 \leftarrow \text{Start}}^{3 \leftarrow \text{END}} k = 1+2+3 = 6$$

k is the index; tells you where to start!
n is where you end!

Properties of Summation Notation

A) $\sum_{k=1}^{n} c \cdot n$ "c" is any constant

$$\sum_{k=1}^{5} 3 = 3+3+3+3+3 = 15$$

c = 3

B) $\sum_{k=1}^{n} c \cdot a_k \qquad c\sum_{k=1}^{n} a_k$

$$\sum_{k=1}^{5} 3k \rightarrow 3(1+2+3+4+5) = 45$$

or

$$3(1) + 3(2) + 3(3) + 3(4) + 3(5) = 45$$

C) $\sum_{k=1}^{n} (a_k + b_k) \rightarrow \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

A) $\sum_{k=1}^{n} c\cdot n$ "c" is any constant

$\sum_{k=1}^{5} 3 = 3+3+3+3+3 = 15$

$c=3$

B) $\sum_{k=1}^{n} c\cdot a_k \quad c\sum_{k=1}^{n} a_k$

$\sum_{k=1}^{5} 3k \rightarrow 3(1+2+3+4+5) = 45$

or

$3(1)+3(2)+3(3)+3(4)+3(5) = 45$

c) $\sum_{k=1}^{n} (a_k+b_k) \rightarrow \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

$\sum_{k=1}^{4} k^2+k \longrightarrow \sum_{k=1}^{4} k^2 + \sum_{k=1}^{4} k \rightarrow \left((1)^2+(2)^2+(3)^2+(4)^2\right) + (1+2+3+4) = 40$

d) $\sum_{k=1}^{n} (a_k-b_k) \rightarrow \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

$\sum_{k=1}^{4} k^2-k \rightarrow \sum_{k=1}^{4} k^2 - \sum_{k=1}^{4} k \rightarrow (1^2+2^2+3^2+4^2) - (1+2+3+4) = 20$

Ex. 6 Evaluate

a) $\sum_{k=1}^{5} 2k$

b) $\sum_{k=1}^{4} k^2+3k+4$

$$2\sum_{k=1}^{5} k$$

$$2(1+2+3+4+5)$$

$$2(15)$$

$$30$$

$$\sum_{k=1}^{4} k^2 + 3\sum_{k=1}^{4} k + \sum_{k=1}^{4} 4$$

$$(1^2+2^2+3^2+4^2)+3(1+2+3+4)+(4+4+4+4)$$

$$76$$

c) $\displaystyle\sum_{k=3}^{6} k^2-4k$

$$\sum_{k=3}^{6} k^2 - 4\sum_{k=3}^{6} k = 14$$

$$(3^2+4^2+5^2+6^2)-4(3+4+5+6)$$

d) Write the sum of $\displaystyle\sum_{k=1}^{n} \frac{k}{4}$

$$\frac{1}{4}+\frac{2}{4}+\frac{3}{4}+\frac{4}{4}+\frac{5}{4}+\dots+\frac{n}{4}$$

Ex.7 Express using Summation Notation

$+ \dots + 9^2$

$$\sum_{k=1}^{9} k^2$$

Ex.8 (Tb pg. 624 #9)

From 1960 to 1997, the resident population of the United States could be approximated by the model:

$$a_n = \sqrt{33{,}282 + 801.3n + 6.12n^2}$$

Where a_n = the population in millions and n represents the calendar year. Find the last five terms of this

Finite Sequence.

$a_{37} = \sqrt{33{,}282 + 801.3(37) + 6.12(37)^2}$

$\approx$ 267 million in 1997

$$\begin{array}{r} 1997 \\ -1960 \\ \hline \end{array}$$

$n = 0, 1, 2, \ldots 37$

$1960 = 0$, $1961 = 1$, $1962 = 2$

$a_{36} = \sqrt{33{,}282 + 801.3(36) + 6.12(36)^2}$

$\approx$ 264.7 million in 1996

$a_{35} =$ " " 262.3 million 1995

$a_{34} =$ " " 260 million 1994

$a_{33} =$ " " 257.7 million 1993