2.3 Real Zeros of Polynomial Functions

Division Algorithm: f(x) = d(x) q(x) + r(x); * $\boxed{d(x) \neq 0}$ *

quotient

dividend divisor remainder

Long Division of Polynomials:

$$\begin{array}{r} 95 \\ 6\overline{)574} \\ -54\downarrow \\ 34 \\ -30 \\ 4 \end{array} \qquad 95\tfrac{4}{6}$$

- Dividend's powers must be in descending order. ✓
- Any missing powers must be replaced with $0x^n$. ✓
- Quotient's powers will line up over the dividend powers. ✓

Ex: $f(x) = 6x^3 - 19x^2 + 16x - 4 \div x - 2$

$$\begin{array}{r} 6x^2 - 7x + 2 \\ x-2\overline{)6x^3 - 19x^2 + 16x - 4} \\ -6x^3 + 12x^2 \downarrow \\ -7x^2 + 16x \\ +7x^2 + 14x \\ 2x - 4 \\ -2x + 4 \\ 0 \end{array}$$

$6x^2 - 7x + 2;\ x \neq 2$

Ex. 1: Divide using Long Division

a) $(2x^2 + 10x + 12) \div (x+3)$

$$\begin{array}{r} 2x + 4 \\ x+3\overline{)2x^2 + 10x + 12} \\ -2x^2 + 6x \downarrow \\ 4x + 12 \\ -4x + 12 \\ 0 \end{array}$$

$2x + 4;\ x \neq -3$

b) $\dfrac{x^3 - 1}{x - 1}$

$$\begin{array}{r} x^2 + x + 1 \\ x-1\overline{)x^3 + 0x^2 + 0x - 1} \\ -x^3 + 1x^2 \downarrow \\ 1x^2 + 0x \\ -1x^2 + 1x \\ 1x - 1 \\ -1x + 1 \\ 0 \end{array}$$

$x^2 + x + 1;\ x \neq 1$

c) $(2x^4 + 4x^3 - 5x^2 + 3x - 2)(x^2 + 2x - 3)^{-1}$

$$\begin{array}{r} 2x^2 \quad + 1 \\ x^2 + 2x - 3\overline{)2x^4 + 4x^3 - 5x^2 + 3x - 2} \\ -2x^4 + 4x^3 + 6x^2 \downarrow \ \downarrow \\ x^2 + 3x - 2 \\ -x^2 + 2x + 3 \\ x + 1 \end{array}$$

$2x^2 + 1 + \dfrac{x+1}{x^2 + 2x - 3};\ x \neq -3, 1$

$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$x = -3 \quad x = 1$

FYI

$(a+b)^{-1} \rightarrow \dfrac{1}{a+b}$

$(2x+3)(x-1)^{-1} \rightarrow (2x+3) \cdot \dfrac{1}{x-1}$

$\dfrac{2x+3}{x-1}$

A shortcut for long division of polynomials that have a divisor in the form **(x-k)** is called **Synthetic Division**.

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Synthetic Division:

- Only works for divisors with linear binomials.
- Dividend's powers must be in descending order.
- Any missing powers must be replaced with $0x^n$.
- Use the coefficients of the dividend.
- Divisor is opposite in ___. ($x+3=0 \rightarrow x=-3$)
- Quotient contains the coefficients of the new polynomial with $n-1$ powers descending.

ex: $(x^4-10x^2-2x+4)\div(x+3)$

$x+3=0$
$x=-3$

$$\begin{array}{r|rrrrr} -3 & 1 & 0 & -10 & -2 & 4 \\ & \downarrow & -3 & 9 & 3 & -3 \\ \hline & 1 & -3 & -1 & 1 & 1 \end{array}$$

$x^3-3x^2-x+1+\frac{1}{x+3};\ x\neq -3$

Ex. 2: Divide using Synthetic Division

a) $(9x^3-18x^2-16x+32)(x-2)^{-1}$

$$\begin{array}{r|rrrr} 2 & 9 & -18 & -16 & 32 \\ & \downarrow & 18 & 0 & 32 \\ \hline & 9 & 0 & -16 & 0 \end{array}$$

$9x^2-16;\ x\neq 2$

b) $\frac{5x^3+6x+8}{x+2}$

$$\begin{array}{r|rrrr} -2 & 5 & 0 & 6 & 8 \\ & \downarrow & -10 & 20 & -52 \\ \hline & 5 & -10 & 26 & -44 \end{array}$$

$5x^2-10x+26-\frac{44}{x+2};\ x\neq -2$

Remainder Theorem:

If a polynomial f(x) is divided by $x-k$, the remainder is **$r = f(k)$**

This means that (k, r) is a point on the graph of the function.

Ex. 3: What is the remainder?

a) $(3x^3+8x^2-7)\div(x+2)$

$f(-2)=3(-2)^3+8(-2)^2-7$

$f(-2)=1$

Remainder of 1; $(-2,1)$ is a point on the graph of the function.

b) $f(x)=3x^3+8x^2+5x-7 \div (x+2)$

$f(-2)=3(-2)^3+8(-2)^2+5(-2)-7$

$f(-2)=-9$

Remainder -9

$(-2,-9)$

2.3 Real Zeros of Polynomial Functions

Factor Theorem:

A polynomial f(x) has a factor (x – k) if and only if **f(k) = 0**

Ex. 4: Show that the given factors are indeed factors of the given polynomial.

a) $(x+3)$ and $(x-2)$; $f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$

$f(-3) = 2(-3)^4 + 7(-3)^3 - 4(-3)^2 - 27(-3) - 18 = 0$ ✓

$f(2) = 2(2)^4 + 7(2)^3 - 4(2)^2 - 27(2) - 18 = 0$ ✓

x+3 and x-2 are factors

b) $(x+4)$; $f(x) = 6x^4 - 8x^3 + 3x^2 - 17$

$f(-4) = 6(-4)^4 - 8(-4)^3 + 3(-4)^2 - 17$

$= 2079$

No b/c r = 2079 not 0.

Ex. 5: Verify the given factors; then find the remaining factors.

a) $f(x) = 8x^4 - 14x^3 - 71x^2 - 10x + 24$; $(x+2)(x-4)$

x+2

$$\begin{array}{r|rrrrr} -2 & 8 & -14 & -71 & -10 & 24 \\ & \downarrow & -16 & 60 & 22 & -24 \\ \hline & 8 & -30 & -11 & 12 & 0 \end{array}$$

x-4

$$\begin{array}{r|rrrr} 4 & 8 & -30 & -11 & 12 \\ & \downarrow & 32 & 8 & -12 \\ \hline & 8 & 2 & -3 & 0 \end{array}$$

Depressed Polynomial $\longrightarrow$ $8x^2 + 2x - 3 = 0$ $\longleftrightarrow$ Factor or Quadratic Formula!

$2x$ | -24: $-1 \cdot 24$, $-2 \cdot 12$, $-3 \cdot 8$, $-4 \cdot 6$

$(8x^2 - 4x)(6x - 3) = 0$

$4x(2x-1) + 3(2x-1)$

$(4x+3)(2x-1) = 0$

$(4x+3)(2x-1)(x+2)(x-4)$

Quadratic Formula

$x = -3/4$ $x = 1/2$

$4x = -3$ $2x = 1$

$4x + 3 = 0$ $2x - 1 = 0$

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Rational Zero Test:
Used to find **POSSIBLE** rational zeros of a polynomial.

Possible rational zeros = $\dfrac{\text{Factors of the CONSTANT TERM}}{\text{Factors of the LEADING COEFF.}}$

ex: $f(x) = 5x^4 - 8x^3 + 3x^2 + 8x + 6$

$PRZ = \dfrac{\overset{\text{constant}}{6}}{\text{LC } 5} \rightarrow \dfrac{\pm1 \pm2 \pm3 \pm6}{\pm1 \pm5} \rightarrow \pm1, \pm2, \pm3, \pm6, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{6}{5}$

Ex. 6: State the possible rational zeros; then find the zeros.

a) $f(x) = 2x^3 + 3x^2 - 8x + 3$

$PRZ = \dfrac{3}{2} \rightarrow \dfrac{\pm1 \pm3}{\pm1 \pm2} \rightarrow \pm1, \pm3, \pm\frac{1}{2}, \pm\frac{3}{2}$ * Start with a integer #!

$x+1$
$$\begin{array}{r|rrrr} -1 & 2 & 3 & -8 & 3 \\ & \downarrow & -2 & -1 & 9 \\ \hline & 2 & 1 & -9 & 12 \end{array}$$

$x+1$ does not work.

$\{-3, 1/2, 1\}$

$x-1$
$$\begin{array}{r|rrrr} 1 & 2 & 3 & -8 & 3 \\ & \downarrow & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

$2x^2 + 5x - 3 = 0$

$(2x^2 - 1x)(6x - 3) = 0$

$x(2x-1) + 3(2x-1) = 0$

$(x+3)(2x-1) = 0$

$$\begin{array}{c|c} 5x & -6 \\ \hline -1x + 6x & -1 \cdot 6 \end{array}$$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Quadratic Formula

b) $f(x) = 10x^3 - 15x^2 - 16x + 12$

$PRZ = \dfrac{12}{10} = \dfrac{\pm1 \pm2 \pm3 \pm4 \pm6 \pm12}{\pm1 \pm2 \pm5 \pm10}$

$\pm1, \pm2, \pm3, \pm4, \pm6, \pm12, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{4}{5}, \pm\frac{6}{5}, \pm\frac{12}{5}$

$\pm\frac{1}{10}, \pm\frac{3}{10}$ * Do not rewrite fractions that can be reduced into smaller listed fractions

Evaluate using Factor theorem

$x-1 = f(1) = -9$ X

$x+1 = f(-1) = 3$ X

$x-2 = f(2) = 0$ ✓

$x-2$
$$\begin{array}{r|rrrr} 2 & 10 & -15 & -16 & 12 \\ & \downarrow & 20 & 10 & -12 \\ \hline & 10 & 5 & -6 & 0 \end{array}$$

$10x^2 + 5x - 6 = 0$

$a=10 \quad b=5 \quad c=-6$

$x = \dfrac{-(5) \pm \sqrt{(5)^2 - 4(10)(-6)}}{2(10)}$

$= \dfrac{-5 \pm \sqrt{265}}{20}$

$\left\{2, \dfrac{-5 \pm \sqrt{265}}{20}\right\}$