12.1 Intro to Limits

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10:24 AM

* A fundamental concept of calculus is **Limits**.

Defn. of a Limit

If $f(x)$ becomes arbitrarily close to an unique number N as x approaches c from either side.

$\lim\limits_{x \to c} f(x) = N$ "The limit of $f(x)$ as x approaches c equals N".

As x gets close to C, but remains unequal to C, the corresponding value of $f(x)$ gets closer to N.

ex: $f(x) = \dfrac{x^2-1}{x-1}$ let $x=1$

$f(1) = \dfrac{0}{0} \leftarrow$ undefined

So instead of $x=1$, use "as x approaches 1".

$f(x) = \dfrac{x^2-1}{x-1}$

x	.25	.50	.75	.90	.99	.999	.9999
$f(x)$	1.25	1.5	1.75	1.9	1.99	1.999	1.9999

As x gets closer to 1, $f(x)$ gets closer to 2.

B/c we can not use $x=1$ (its undefined) but it appears that $f(x)$ is going to be 2.

$\lim\limits_{x \to 1} f(x) = 2$ or $\lim\limits_{x \to 1} \dfrac{x^2-1}{x-1} = 2$

Ex.1 Find the Limit Using a table

a) $\lim\limits_{x \to 3} 5x^2$

x	2	2.5	2.9	2.99	2.999	3	3.0001	3.001	3.01	3.1	3.5
$f(x)$	20	31.25	42.05	44.7005	44.910005	—	45.003	45.0300	45.030	45.30	48.05

$\lim\limits_{x \to 3} f(x) = 45$

b) $\lim_{x\to 2} \frac{x^2-4}{x-2}$

x	1.9	1.99	1.999	2	2.001	2.01	2.1
f(x)	3.9	3.99	3.999	—	4.001	4.01	4.1

$f(2) = \frac{(2)^2-4}{2-2}$

$f(2) =$ undefined

$\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$

c) $\lim_{x\to 2} \frac{x^3-2x^2+4x-8}{x^4-2x^3+x-2}$

x	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
f(x)	.9683	.8964	.8896	.8889	—	.8888	.8882	.8815	.8196

$\lim_{x\to 2} f(x) = .8888$

Finding limits of graphs that are continuous (above examples) is easier than noncontinuous graphs.

d) $\lim_{x\to 0} \frac{x}{\sqrt{x+1}-1}$

x	-0.01	-0.001	-0.0001	0	0.0001	0.001	.01
f(x)	1.995	1.9995	1.99995	—	2.00005	2.0005	2.005

$x = 0$ is undefined

$\lim_{x\to 0} f(x) = 2$

e) $\lim_{x\to 0} \frac{\sin x}{x}$

x	-0.3	-0.2	-0.1	-0.01	-0.001	0	.001	.01	.1	.2
f(x)	.9851	.9934	.9983	.99998	.999999998	—	.999999998	.99998	.9983	.9834

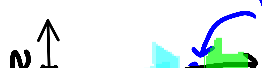
$\lim_{x\to 0} \frac{\sin x}{x} = 1$

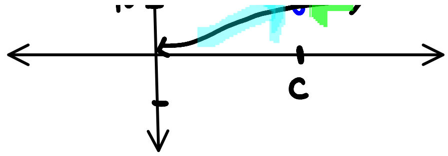
Some functions can have a limit even though the number that it approaches makes f(x) undefined.

* It is important to realize that the existence or nonexistence of f(x) when $x = c$ has no bearing on the existence of the limit of f(x) as x approaches c.

You can use graphs to determine limits!

hole in graph; however, it might not have a hole!



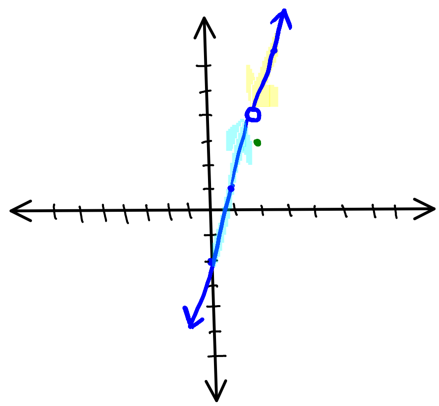


$f(c) = N$

$\lim_{x \to c} f(x) = N$

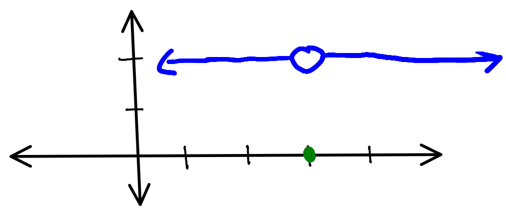
Ex. 2 Find the limit of f(x) as x approaches 2 where

a) $f(x) = \begin{cases} 3x-2, & x \neq 2 \\ 3, & x = 2 \end{cases}$



$\lim_{x \to 2} f(x) = 4$

b) $f(x) = \begin{cases} 2, & x \neq 3 \\ 0, & x = 3 \end{cases}$



$\lim_{x \to 2} f(x) = 2$

$\lim_{x \to \#^-} f(x)$ from left

$\lim_{x \to \#^+} f(x)$ from right

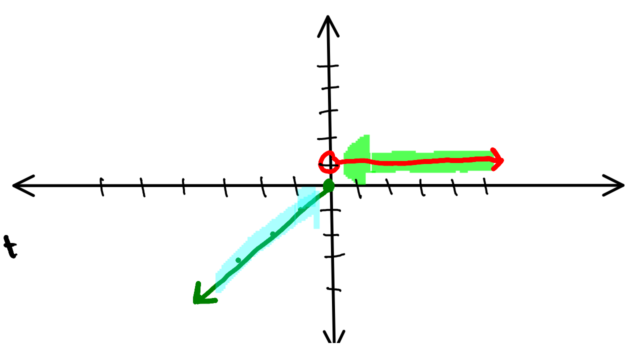
Limits that Fail to Exist

Sometimes there is no single # that the value of "f" approaches as x approaches c. If this occurs, f has no limit as x approaches c.

$\lim_{x \to c} f(x) =$ does not exist

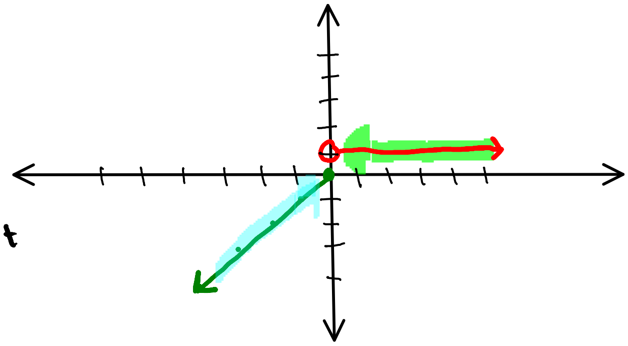
Ex. 3 Find the limit if it exists.

a) $f(x) = \begin{cases} x, & x \leq 0 \\ 1, & x > 0 \end{cases}$



as x approaches 0 from left side (neg) lim = 0

a) $f(x) = \begin{cases} x, & x \le 0 \\ 1, & x > 0 \end{cases}$



as x approaches 0 from left side (neg) lim = 0

as x approaches 0 from right side (pos) lim = 1

$\lim_{x \to 0} f(x)$ = Does not exist

b) $\lim_{x \to 0} \frac{1}{x}$

x	-0.5	-0.2	-0.1	-0.01	0	0.01	0.1	0.2	0.5
f(x)	-2	-5	-10	-100	—	100	10	5	2

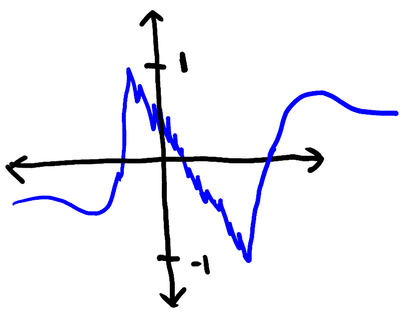
$\lim_{x \to 0} \frac{1}{x}$ = does not exist

This function displays "Unbounded Behavior"

as x approaches 0 from right, f(x) approaches ∞

as x approaches 0 from left, f(x) approaches $-\infty$

c) $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$



$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ = does not exist.

Oscillating Behavior

Conditions Under Which Limits Do Not Exist

The limit of f(x) as $x \to c$ does not exist if any of the following conditions are true:

1) f(x) approaches a different number from the right of c than the left of c. (ex. 3a)

2) f(x) increases or decreases without bound as x approaches c. (ex. 3b)

3) f(x) oscillates between 2 fixed values as x approaches c (ex. 3c)

Properties of Limits

Sometimes the limit of $f(x)$ as $x \to c$ is just $f(c)$. This is Direct Substitution.

$$\lim_{x \to c} f(x) = f(c)$$

Functions that are like the above statement are called "Well behaved" and its limit can be evaluated by direct substitution of c into $f(x)$.

$$\lim_{x \to 5} 2x+1 \to f(5) = 2(5)+1 = 11 \qquad \lim_{x \to 5} 2x+1 = 11 \checkmark$$

$$\lim_{x \to 0} \frac{2x+1}{x} \quad \text{not well behaved!}$$

$$\lim_{x \to c} b = b \qquad \lim_{x \to c} x = c \qquad \lim_{x \to c} x^n = c^n \qquad \lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$$

$$\lim_{x \to 3} 5 = 5 \qquad \lim_{x \to 3} x = 3 \qquad \lim_{x \to 4} x^2 = 16 \qquad \lim_{x \to 27} \sqrt[3]{x} = \sqrt[3]{27} = 3$$

$$\lim_{x \to c} \sin x = \sin c$$

$$\lim_{x \to \pi} \sin x = \sin \pi = 0$$

Operations with Limits

Let $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = K$

Scalar Multiplication

$$\lim_{x \to c} [b f(x)] = bL$$

sum/difference

$$\lim_{x \to c} [f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$$

Product

$$\lim_{x \to c} [f(x) g(x)] = \left[\lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)\right]$$

Quotient

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$$

Power

$$\lim [f(x)]^n = \left[\lim f(x)\right]^n$$

$$\lim_{x\to c}[f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n$$

Ex.4 Find the limit (if it exists) using direct substitution.

a) $\lim_{x\to 2} x^2$

$(2)^2$

4

b) $\lim_{x\to 4} 5$

5

c) $\lim_{x\to 4} (x^2+2x-5)$

$(4)^2+2(4)-5$

19

d) $\lim_{x\to 4} \frac{x^2+2x-5}{x-1}$

$\frac{(4)^2+2(4)-5}{4-1}$

$\frac{19}{3}$

e) $\lim_{x\to 1/2} x$

$\frac{1}{2}$

f) $\lim_{x\to 0} \cos x$

$\cos 0$

1

g) $\lim_{x\to 2} (-4x^3)$

$(-4(2)^3)$

-32

h) $\lim_{x\to 0} \sqrt{5x^2+8}$

$\sqrt{5(0)^2+8}$

$\sqrt{8}$

$2\sqrt{2}$

j) $\lim_{x\to -1} (5x^3-x+3)^{4/3}$

$(5(-1)^3-(-1)+3)^{4/3}$

1