3.5 – Finding (All) Zeros of Polynomials

There are 2 types of zeros for polynomial functions:

– **Real Zero** → any type of zero that does not contain "i" (imaginary)
* These zeros cross or touch the x-axis!

- **rational zero(s)** – zeros that are Whole #'s, terminating decimals (.25, .5, .125....), or repeating decimals (.33333..., .6666666...) which can be written as a FRACTIONS (I call these "pretty" zeros).

- **irrational zero** – zeros that are non-terminating decimals, non-repeating decimals, or contain square roots. * These zeros cannot be written as decimals.

– **Imaginary Zero** → zeros that contain the imaginary unit of i. This is where these zeros DO NOT touch the x-axis. These types of zeros **COME IN PAIRS**! $x = \pm i$

Convert ALL DECIMALS TO FRACTIONS

Common Rational Zeros (aka fractions)	
0.25 →	1/4
$0.\overline{3}$ →	1/3
0.5 →	1/2
$0.\overline{6}$ →	2/3
0.75 →	3/4
$1.\overline{3}$ →	1 1/3 → 4/3
1.5 →	1 1/2 → 3/2
$1.\overline{6}$ →	1 2/3 → 5/3
2.5 →	2 1/2 → 5/2
3.5 →	3 1/2 → 7/2

Steps to Find All Zeros of a Polynomial

1.) Put polynomial function into a graphing calculator:

Ti-83+ or higher

a. Put polynomial P(x) in Y1 = and have Y2 = 0, find ANY or ALL rational zero(s).

b. Use 2nd Trace: #5:Intersection (this is easier than #2:Zero).Hit ENTER 3 times. Do this for all the real zeros you see; move cursor on Y1 each time closer to the next x-intercept.

DEMOS "Online"

a. Put polynomial P(x) into the function box 1.

b. Zoom in or out to see the points where the graph crosses or touches the x-axis. Find ANY or ALL rational zero(s).

2.) Use synthetic division to divide any rational zero into the polynomial P(x) (remainder should = 0). The zero found in Step # 1 will = c … **so don't** change the sign in the half box when doing synthetic ÷. * You may have to do LONG DIVISION *

3.) Repeat Step # 2, using continuous synthetic division, until have used all rational zeros for P (x). The goal is to get a quotient down to either a LINEAR EQUATION or QUADRATIC EQUATION.

For quadratic equations you will have to use the Quadratic Formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

4.) When listing all zeros remember… degree of polynomial P (x) = # of zeros for polynomial P (x).
A method of checking YOURSELF!

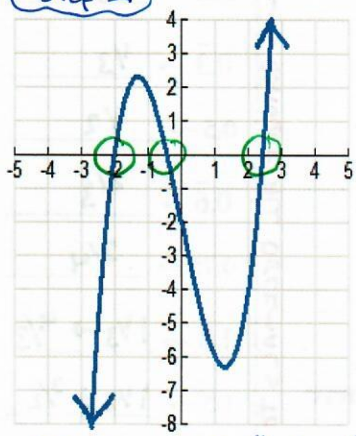
* Remember Zeros can have multiplicity *

Example 1: Using the given polynomial P (x) and its graph, find all the zeros for P (x).

Will have 3 zeros; graph indicates ALL REAL!

a.) $P(x) = x^3 - 5x - 2$ → all zeros of P (x) = $-2, 1 \pm \sqrt{2}$

Step 1



* Start with the "integer" Real zero!

Step 2

missing degree of 2!

$-2 \,|\; 1 \quad 0 \quad -5 \quad -2$

$\quad\; \downarrow \quad -2 \quad 4 \quad 2$

$\quad\; 1 \quad -2 \quad -1 \;|\; 0$ ← should be zero!

$x^2 - 2x - 1 = 0$

Cannot Factor!

Use quadratic formula!

Step 3

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a = 1 \quad b = -2 \quad c = -1$

$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$

$x = \frac{2 \pm \sqrt{8}}{2}$ → $\sqrt{8}$: 2, 4 → 2, 2 → $2\sqrt{2}$

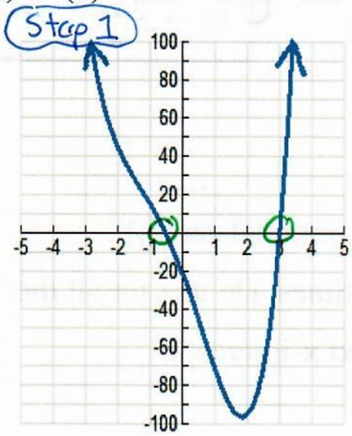
$x = \frac{2 \pm 2\sqrt{2}}{2}$

$x = 1 \pm \sqrt{2}$ Give EXACT ANSWER!

Will have 4 zeros; graph indicates 2 REAL and 2 imaginary!

b.) $P(x) = 2x^4 + 3x^3 - 9x^2 - 47x - 21$ → all zeros of P (x) = $-1/2, 3, -2 \pm i\sqrt{3}$

Step 1



* Real zeros $-1/2$, 3

Step 2

$3 \,|\; 2 \quad 3 \quad -9 \quad -47 \quad -21$

$\quad\; \downarrow \quad 6 \quad 27 \quad 54 \quad 21$

$\quad\; 2 \quad 9 \quad 18 \quad 7 \;|\; 0$

$2x^3 + 9x^2 + 18x + 7$

Do division again

$x = -\frac{1}{2}$

$-\frac{1}{2} \,|\; 2 \quad 9 \quad 18 \quad 7$

$\quad\; \downarrow \quad -1 \quad -4 \quad -7$

$\quad\; 2 \quad 8 \quad 14 \;|\; 0$

Step 3

$2x^2 + 8x + 14 = 0$ FACTOR OUT A "2" to make #'s smaller!

$x^2 + 4x + 7 = 0$

$a = 1 \quad b = 4 \quad c = 7$

$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(7)}}{2(1)}$

$x = \frac{-4 \pm \sqrt{-12}}{2}$

indicates there will be an imaginary solution!

$x = \frac{-4 \pm \sqrt{-12}}{2}$ → $\sqrt{-12}$ → $\sqrt{-1} \cdot \sqrt{12}$ → i; 2, 6 → 2, 3

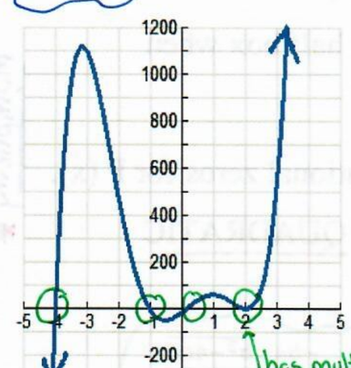
$x = \frac{-4 \pm 2i\sqrt{3}}{2}$

$x = -2 \pm i\sqrt{3}$

Will have 5 zeros; graph indicates there will be 5 REAL ZEROS

c.) $P(x) = 7x^5 + 6x^4 - 85x^3 + 40x^2 + 108x - 16$ → all zeros of P (x) = $-4, -1, 2$ (mo2), $1/7$

Step 1



has multiplicity of 2!

REAL zeros → $-4, -1, 2, 2$

Step 2

$-4 \,|\; 7 \quad 6 \quad -85 \quad 40 \quad 108 \quad -16$

$\quad\; \downarrow \quad -28 \quad 88 \quad -12 \quad -112 \quad 16$

$\quad\; 7 \quad -22 \quad 3 \quad 28 \quad -4 \;|\; 0$

* Step 2 again! *

$-1 \,|\; 7 \quad -22 \quad 3 \quad 28 \quad -4$

$\quad\; \downarrow \quad -7 \quad 29 \quad -32 \quad 4$

$\quad\; 7 \quad -29 \quad 32 \quad -4 \;|\; 0$

* Step 2 again! *

$2 \,|\; 7 \quad -29 \quad 32 \quad -4$

$\quad\; \downarrow \quad 14 \quad -30 \quad 4$

$\quad\; 7 \quad -15 \quad 2 \;|\; 0$

* Step 2 one more time! *

$2 \,|\; 7 \quad -15 \quad 2$

$\quad\; \downarrow \quad 14 \quad 2$

$\quad\; 7 \quad -1 \;|\; 0$

Step 3

$7x - 1 = 0$

$7x = 1$

$x = 1/7$

Example 2: Find all the zeros of each polynomial P (x).

a.) $P(x) = x^3 + 2x^2 - 2x - 1$ → Indicates 3 zeros!

Step 1: Graph and find REAL zeros

Follow steps from 1st page!

$x = 1$

Step 2

1	1	2	-2	-1
	↓	1	3	1
	1	3	1	0

$x^2 + 3x + 1 = 0$

Cannot factor, use quadratic formula!

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a = 1 \quad b = 3 \quad c = 1$

Step 3

$x = \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(1)}}{2(1)}$

$x = \frac{-3 \pm \sqrt{5}}{2}$ ← Exact Solution; No decimals!

All Zeros of P (x) = $1, \frac{-3 \pm \sqrt{5}}{2}$

b.) $P(x) = 3x^3 + 14x^2 + 23x + 10$ — Indicates 3 zeros!

Step 1 Graph and find REAL ZEROS

Real zero → $-2/3$ Imaginary → 2!

Step 2

$-2/3$	3	14	23	10
	↓	-2	-8	-10
	3	12	15	0

$3x^2 + 12x + 15 = 0$

Step 3 FACTOR OUT A 3 First!

$x^2 + 4x + 5 = 0$

$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(5)}}{2(1)}$

$x = \frac{-4 \pm \sqrt{-4}}{2}$

$x = \frac{-4 \pm \sqrt{-4}}{2}$

$\sqrt{-4} = \sqrt{-1} \cdot \sqrt{4} = i \cdot 2$

$x = \frac{-4 \pm 2i}{2}$

$x = -2 \pm i$

All Zeros of P (x) = $-2/3, -2 \pm i$

c.) $P(x) = 2x^4 - 10x^3 + 3x^2 + 36x - 27$ — 4 zeros

Step 1: Graph and find REAL ZEROS!

4 real zeros — 2 rational, 2 irrational

$x = 3$ (mo 2)

Step 2

3	2	-10	3	36	-27
	↓	6	-12	-27	27
	2	-4	-9	9	0

3	2	-4	-9	9
	↓	6	6	9
	2	2	-3	0

$2x^2 + 2x - 3 = 0$

Quadratic Formula!

Step 3

$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(2)(-3)}}{2(2)}$

$x = \frac{-2 \pm \sqrt{28}}{4}$

$\sqrt{28}$: $2 \cdot 14$, $2 \cdot 7$

$x = \frac{-2 \pm 2\sqrt{7}}{4}$

$x = \frac{-1 \pm \sqrt{7}}{2}$

All Zeros of P (x) = 3 (mo 2), $\frac{-1 \pm \sqrt{7}}{2}$

d.) $P(x) = 3x^5 + 7x^4 + x^3 - 17x^2 - 30x$ — 5 zeros!

Step 1 Graph and Find Real zeros!

3 Real zeros: $-2, 0, 5/3$; 2 imaginary zeros!

Step 2

missing the constant!

-2	3	7	1	-17	-30	0
	↓	-6	-2	2	30	0
	3	1	-1	-15	0	0

0	3	1	-1	-15	0
	↓	0	0	0	0
	3	1	-1	-15	0

5/3	3	1	-1	-15
	↓	5	10	15
	3	6	9	0

$3x^2 + 6x + 9 = 0$

FACTOR 3 OUT!

$x^2 + 2x + 3 = 0$

Step 3

$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(3)}}{2(1)}$

$x = \frac{-2 \pm \sqrt{-8}}{2}$

$\sqrt{-8}$: $\sqrt{-1} = i$, $\sqrt{8}$: $2 \cdot 4$, $2 \cdot 2$

$x = \frac{-2 \pm 2i\sqrt{2}}{2}$

$x = -1 \pm i\sqrt{2}$

All Zeros of P (x) = $-2, 0, 5/3, -1 \pm i\sqrt{2}$