3.4 – Solving Polynomials: Quadratic Techniques and R/F Theorems

Solving Polynomials Using Quadratic Techniques

– **Quadratic form** → an expression that can be written as $au^2 + bu + c = 0$ for any numbers a, b, and c as long as $a \neq 0$ where "u" is some expression in terms of "x"

ie: $x^4 - 7x^2 - 18$ ---> $u = x^2$ --> substitue u in → $u^2 - 7u - 18$

If you have the FACTORS of a polynomial, then you already have the zeros / roots for that polynomial.

ie.: $\underbrace{x^2 + 5x + 6}_{\text{FACTOR}} = 0$

To find the zeros / roots use the zero product property, set each factor equal to 0.

Factors → $(x+2)(x+3) = 0$

$x+2=0 \qquad x+3=0$

Roots → $x=-2 \qquad x=-3$ → Solution Set $\{-3, -2\}$

Use the Quadratic Formula to solve quadratics that cannot be factored.

$ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{(b)^2 - 4ac}}{2a}$$ only use if you cannot factor!

$\sqrt{-4} \to \pm 2i$

*Remember Imaginary Solutions come in PAIRS ± ; *

Example 1: Solve by factoring and other quadratic techniques.

a.) $x^4 - 7x^2 - 18 = 0$	b.) $2x^2 - 8x + 3x - 12 = 0$	c.) $x^4 - 16 = 0$	d.) $x^3 + 2x^2 - 3x = 0$
Should have 4 solutions *FACTOR* $x^4 \to x^2 \cdot x^2$ ① -18 a·c: $2x^2 - 9x^2$ 1·-18 ✗ 2·-9 ✓ 3·-6 ✗ bx: $-7x^2$ $x^2 \quad -9$ x^2: $x^4 \quad -9x^2$ $+2$: $2x^2 \quad -18$ ② $(x^2+2)(x^2-9) = 0$ $x^2+2=0 \quad x^2-9=0$ $x^2=-2 \quad x^2=9$ $\sqrt{x^2}=\sqrt{-2} \quad \sqrt{x^2}=\sqrt{9}$ $x=\pm i\sqrt{2} \quad x=\pm 3$ $\{-3, 3, \pm i\sqrt{2}\}$	*FACTOR BY GROUPING* ① $(2x^2-8x)(+3x-12) = 0$ $2x(x-4) + 3(x-4) = 0$ $(2x+3)(x-4) = 0$ ② solve each factor $2x+3=0 \quad x-4=0$ $2x=-3 \quad x=4$ $x=-\frac{3}{2}$ $\{-3/2, 4\}$	4 zeros/Roots *Factor "Difference of Squares"* ① $x^4 - 16 = 0$ $(x^2+4)(x^2-4) = 0$ ② Solve each factor! $x^2+4=0 \quad x^2-4=0$ $x^2=-4 \quad x^2=4$ $\sqrt{x^2}=\sqrt{-4} \quad \sqrt{x^2}=\sqrt{4}$ $x=\pm 2i \quad x=\pm 2$ $\{-2, 2, \pm 2i\}$	3 zeros/Roots *Factor a GCF* ① $x(\underbrace{x^2+2x-3}_{\text{Factor}}) = 0$ $x(x+3)(x-1) = 0$ ② Solve for each factor! $x=0 \quad x+3=0 \quad x-1=0$ $x=-3 \quad x=1$ $\{-3, 0, 1\}$

– **Sum / Difference of Cubes** → 1, 8, 27, 64, 125, 216, … Perfect "cubes"

$(1)^3$ $(2)^3$ $(3)^3$ $(4)^3$ $(5)^3$ $(6)^3$

SUM of Cubes $a^3 + b^3 = 0$

DIFFERENCE of Cubes $a^3 - b^3 = 0$

S O A P
Same, Opposite, Always Positive

*Use to write the signs (+, −) between terms of the factored cubed!

Factoring the Sum / Diffrence of CUBES: ⟶ (Binomial)(Trinomial) = 0 — Factored Form

$$a^3 + b^3 = 0 \longrightarrow (a + b)(a^2 - ab + b^2)$$

$\sqrt[3]{a^3} = a$ $\sqrt[3]{b^3} = b$ Same Opposite Always Positive

$$a^3 - b^3 = 0 \longrightarrow (a - b)(a^2 + ab + b^2)$$

Example 2: Solve by factoring and other quadratic techniques.

a.) $x^3 + 64 = 0$ S O AP $(x + 4)((x)^2 - x(4) + (4)^2)$

$\sqrt[3]{x^3} \to x$ (a) $\sqrt[3]{64} \to 4$ (b)

① $(x+4)(x^2 - 4x + 16) = 0$

② Solve factors!

$x + 4 = 0$
$x = -4$

$x^2 - 4x + 16 = 0$ ← CANNOT BE FACTORED!

$a = 1$ $b = -4$ $c = 16$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(16)}}{2(1)}$$

$$x = \frac{4 \pm \sqrt{-48}}{2} \to x = \frac{4 \pm 4i\sqrt{3}}{2}$$

$$x = 2 \pm 2i\sqrt{3}$$

$\sqrt{-48} \to \sqrt{-1} \cdot \sqrt{48}$
i
$48 = 2 \cdot 24$, $24 = 2 \cdot 12$, $12 = 2 \cdot 6$, $6 = 2 \cdot 3$
$2 \cdot 2 \cdot i\sqrt{3}$
$\pm 4i\sqrt{3}$

$\{-4,\ 2 \pm 2i\sqrt{3}\}$

b.) $125y^3 - 27 = 0$ S O AP $(5y - 3)((5y)^2 + 5y(3) + (3)^2) = 0$

$\sqrt[3]{125y^3} \to 5y$ "a"
$\sqrt[3]{27} \to 3$ "b"

① $(5y - 3)(25y^2 + 15y + 9) = 0$

② Solve each factor

$5y - 3 = 0$
$5y = 3$
$y = 3/5$

$25y^2 + 15y + 9 = 0$ cannot be factored
a b c

$$x = \frac{-(15) \pm \sqrt{(15)^2 - 4(25)(9)}}{2(25)}$$

$$x = \frac{-15 \pm \sqrt{-675}}{50}$$

$$x = \frac{-15 \pm 15i\sqrt{3}}{50}$$

$$x = \frac{-3 \pm 3i\sqrt{3}}{10}$$

$\sqrt{-675}$
$\sqrt{-1} \cdot \sqrt{675}$
$i \cdot$ $675 = 3 \cdot 225$, $225 = 3 \cdot 75$, $75 = 3 \cdot 25$, $25 = 5 \cdot 5$
$3 \cdot 5 \cdot i\sqrt{3}$
$\to 15i\sqrt{3}$

$\left\{\frac{3}{5},\ \frac{-3 \pm 3i\sqrt{3}}{10}\right\}$

Solving Polynomials Using Remainder Theorem and Factor Theorem

– **Remainder Theorem** → IF a polynomial $P(x) \div (x - c)$, then the Remainder is equal to $P(c)$. "Polynomial $P(x)$ evaluated at "c" will be the remainder.

ie: $(x^4 - 7x^2 - 18) \div (x+3)$
$x = -3$
$P(-3) = (-3)^4 - 7(-3)^2 - 18$
$P(-3) = 0$ ← Remainder

– **Factor Theorem** → A Polynomial $P(x)$ has a factor $(x - c)$, if and only if $P(c) = 0$;

* IF you evaluate $P(x)$ at "c" and you get 0, then $x - c$ is a factor

ie: Is $(x+2)$ a factor of $(3x^2 - 6x - 24)$? $x + 2 = 0$, $x = -2$ (c)

$P(-2) = 3(-2)^2 - 6(-2) - 24$

$P(-2) = 0$ ← remainder is 0, so $(x+2)$ is a factor!

Example 3: Prove through synthetic division that remainder left and Remainder Theorem equal.

a.) $P(x) = x^3 - 2x^2 + 4x - 1$ is divided by $(x - 5)$

Synthetic Division	Remainder Theorem
$x-5=0 \rightarrow x=5$ $5\rfloor\ 1\ \ -2\ \ 4\ \ -1$ $\downarrow\ \ 5\ \ 15\ \ 95$ $1\ \ 3\ \ 19\ \vert\ 94$ ← remainder * since the remainder is NOT 0, (x-5) IS NOT a factor of x^3-2x^2+4x-1.	$x-5=0 \rightarrow x=5$ $P(5)=(5)^3-2(5)^2+4(5)-1$ $=125-50+25-1$ $P(5)=94$ ← not zero FASTER!

missing x^2 and x!

b.) $P(x) = 2x^4 + 3x^3 - 3$ is divided by $(x + 1)$

Synthetic Division	Remainder Theorem
$x+1=0 \rightarrow x=-1$ $-1\rfloor\ 2\ \ 3\ \ 0\ \ 0\ \ -3$ $\downarrow\ \ -2\ \ -1\ \ 1\ \ -1$ $2\ \ 1\ \ -1\ \ 1\ \vert\ -4$ remainder	$x+1=0 \rightarrow x=-1$ $P(-1)=2(-1)^4+3(-1)^3-3$ $=2(1)+3(-1)-3$ $=2-3-3$ $P(-1)=-4$

* since the remainder is NOT 0, (x+1) IS NOT a factor of $2x^4+3x^3-3$!

Example 4: Find the value(s) of k so that each remainder is 3.

a.) $(x^2+kx-17)\div(x-2)$	b.) $(x^2+5x+7)\div(x+k)$	c.) $(x^3+4x^2+x+k)\div(x+2)$
* use the REMAINDER Thm to find value of "k". ① $x-2=0 \rightarrow x=2$ ② $P(2)=3$ ← remainder from directions ③ $x^2+kx-17=3$ $(2)^2+k(2)-17=3$ $4+2k-17=3$ ← solve for "k" $2k-13=3$ $2k=16$ $k=8$	* use the Remainder thm * ① $x+k=0 \rightarrow x=-k$ ② $P(-k)=3$ ③ $(-k)^2+5(-k)+7=3$ $k^2-5k+7=3$ $k^2-5k+4=0$ ← Factor! $(k-4)(k-1)=0$ $k-4=0\quad k-1=0$ $k=4\quad k=1$	① $x+2=0 \rightarrow x=-2$ ② $P(-2)=3$ ③ $x^3+4x^2+x+k=3$ $(-2)^3+4(-2)^2+(-2)+k=3$ $-8+16-2+k=3$ $6+k=3$ $k=-3$

Example 5: Complete each application appropriately.

a.) If $P(x) = 2x^3 + 3x^2 - 11x - 6$ where $(x - 2)$ and $(x + 3)$ are factors of P (x), then find the remaining factor of P (x).

should have 3 factors!

① Synthetic Division x 2

$x-2=0 \qquad x+3=0$

$x=2 \qquad x=-3$

$2\rfloor\ 2\ \ 3\ \ -11\ \ -6$

$\downarrow\ \ 4\ \ 14\ \ 6$

$2\ \ 7\ \ 3\ \vert\ 0$

"Depressed polynomial" → $2x^2+7x+3$

$-3\rfloor\ 2\ \ 7\ \ 3$

$\downarrow\ \ -6\ \ -3$

$2\ \ 1\ \vert\ 0$

$2x+1$

remainders should be zero!

(2x+1) is the remaining factor!

b.) If $P(x) = 3x^4 + 2x^3 - 36x^2 - 72x - 32$ here $(3x + 2)$ and $(x + 2)^2$ are factors, then find the remaining zero.

4 factors/zeros!

① Do synthetic division with (x+2) and (x+2)

$-2\rfloor\ 3\ \ 2\ \ -36\ \ -72\ \ -32$

$\downarrow\ \ -6\ \ 8\ \ 56\ \ 32$

$3\ \ -4\ \ -28\ \ -16\ \vert\ 0$

$3x^3-4x^2-28x-16$

$-2\rfloor\ 3\ \ -4\ \ -28\ \ -16$

$\downarrow\ \ -6\ \ 20\ \ 16$

$3\ \ -10\ \ -8\ \vert\ 0$

$3x^2-10x-8$

② Do Long division with 3x+2

$$\begin{array}{r} x-4 \\ 3x+2\,\overline{)\,3x^2-10x-8} \\ -(3x^2+2x) \\ \hline -12x-8 \\ -(-12x-8) \\ \hline 0 \end{array}$$

X-4 is the remaining factor!

Should be 4 zeros!

c.) If $P(x) = x^4 - 2x^3 - 20x^2 + 32x + 64$ where its zeros are ± 4 then find remaining zeros.

① Synthetic division twice, -4 and 4.

$$\begin{array}{r|rrrrr} -4 & 1 & -2 & -20 & 32 & 64 \\ & \downarrow & -4 & 24 & -16 & -64 \\ \hline 4 & 1 & -6 & 4 & 16 & 0 \\ & \downarrow & 4 & -8 & -16 & \\ \hline & 1 & -2 & -4 & 0 & \end{array}$$

$x^2 - 2x - 4 = 0$ ← can not factor use quadratic formula.

$a = 1 \quad b = -2 \quad c = -2$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{20}}{2}$$

$\sqrt{20}$: 2, 10; 2, 5

$$= \frac{2 \pm 2\sqrt{5}}{2}$$

$x = 1 \pm \sqrt{5}$

$1 \pm \sqrt{5}$ is the remaining zeros.

d.) The volume of a rectangular prism is represented by the expression $x^3 - 11x^2 + 38x - 40$. The length of the prism is $(x - 2)$ and the width is $(x - 4)$. What is the height of the prism? * Looking for a factor! *

* use synthetic division and the given factors $(x-2)$ and $(x-4)$ to find height.

$V = lwh$

$V = x^3 - 11x^2 + 38x - 40$

$l = (x-2) \rightarrow x = 2$

$w = (x-4) \rightarrow x = 4$

$h = ?$

$$\begin{array}{r|rrrr} 2 & 1 & -11 & 38 & -40 \\ & \downarrow & 2 & -18 & 40 \\ \hline & 1 & -9 & 20 & 0 \end{array}$$

$$\begin{array}{r|rrr} 4 & 1 & -9 & 20 \\ & \downarrow & 4 & -20 \\ \hline & 1 & -5 & 0 \end{array}$$

$x - 5$

$x - 5$ is the height of the prism.